

ON VECTOR MEASURES AND EXTENSIONS OF TRANSFUNCTIONS

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ABSTRACT. We are interested in extending operators defined on positive measures, called here transfunctions, to signed measures and vector measures. Our methods use a somewhat nonstandard approach to measures and vector measures. The necessary background, including proofs of some auxiliary results, is included.

1. INTRODUCTION

By a transfunction [3] we mean a map between sets of measures on measurable spaces. More precisely, if (X, Σ_X) and (Y, Σ_Y) are measurable spaces and $\mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$ and $\mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$ are the sets of all finite positive measures on Σ_X and on Σ_Y , respectively, by a *transfunction* from (X, Σ_X) to (Y, Σ_Y) we mean a map

$$\Phi : \mathcal{M}(X, \Sigma_X, \mathbb{R}^+) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+).$$

If necessary, a transfunction can be defined on a subset of $\mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$.

If $f : (X, \Sigma_X) \rightarrow (Y, \Sigma_Y)$ is a measurable function, then

$$\Phi_f(\mu)(B) = \mu(f^{-1}(B)),$$

for $\mu \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$ and $B \in \Sigma_Y$, defines a transfunction from $\mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$ to $\mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$. If a transfunction Φ is of this form, then clearly the function can be reconstructed from the action of Φ .

If $X = Y = [0, 1]$, Σ is the σ -algebra of Lebesgue measurable subsets of $[0, 1]$, and λ is the Lebesgue measure on $[0, 1]$, then

$$\Phi(\mu) = \mu([0, 1])\lambda$$

defines a transfunction that does not correspond to a function. It can be thought of as a “map” that distributes every input uniformly on $[0, 1]$.

We are interested in transfunctions as a generalization of a function from X to Y . Instead of mapping a point $x \in X$ to a point $y \in Y$ a

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transfunction can be thought of as mapping a probability distribution of the input to a probability distribution of the output. Our long term goal is to investigate to what extent the tools developed for functions can be extended to transfunctions.

We consider the following properties of transfunctions:

Weakly additive: $\Phi(\mu_1 + \mu_2) = \Phi(\mu_1) + \Phi(\mu_2)$ for all $\mu_1, \mu_2 \in \mathcal{M}$ such that $\mu_1 \perp \mu_2$,

Strongly additive: $\Phi(\mu_1 + \mu_2) = \Phi(\mu_1) + \Phi(\mu_2)$ for all $\mu_1, \mu_2 \in \mathcal{M}$,

Homogeneous: $\Phi(\alpha\mu) = \alpha\Phi(\mu)$ for any $\alpha > 0$,

Monotone: $\Phi(\mu_1) \leq \Phi(\mu_2)$, if $\mu_1 \leq \mu_2$,

Norm preserving: $\|\Phi(\mu)\| = \|\mu\|$,

Bounded: $\|\Phi(\mu)\| \leq C\|\mu\|$ for some $C > 0$ and all $\mu \in \mathcal{M}$,

Continuous with respect to weak convergence: If $\mu_n(A) \rightarrow \mu(A)$ for every $A \in \Sigma_X$, then $\Phi(\mu_n)(B) \rightarrow \Phi(\mu)(B)$ for every $B \in \Sigma_Y$.

Continuous with respect to uniform convergence: If $\|\mu_n - \mu\| \rightarrow 0$, then $\|\Phi(\mu_n) - \Phi(\mu)\| \rightarrow 0$.

The definition of transfunctions is very general and it makes sense if finite positive measures are replaced by signed measures of bounded variation or by vector valued measures of bounded variation. Moreover, properties of transfunctions listed above, with the exception of monotonicity, make sense in this general setting. In the last section of this paper we define extensions of transfunctions to signed measures and vector measures. We also discuss the question of uniqueness of such extensions.

The next three sections contain the necessary background and proofs of some auxiliary results.

2. MEASURE SPACES

In this section we recall some definitions and results from [2]. While the results are standard, the approach in [2] is different.

Let X be a nonempty set. A collection \mathcal{R} of subsets of X is called a *ring of subsets* of X if $A, B \in \mathcal{R}$ implies $A \cup B, A \setminus B \in \mathcal{R}$. A map $\mu : \mathcal{R} \rightarrow [0, \infty)$ is called *σ -additive* if for any sequence of disjoint sets $A_1, A_2, \dots \in \mathcal{R}$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$ we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

In this note we use the same symbol to denote a subset of X and the characteristic function of that set, that is, if $A \subset X$ we will write

$$A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}.$$

Let μ be a σ -additive measure on a ring \mathcal{R} of subsets of X . A set $S \subset X$ is called μ -summable if there are $A_1, A_2, \dots \in \mathcal{R}$ and $\alpha_1, \alpha_2, \dots \in \{-1, 1\}$ such that

- I $\sum_{n=1}^{\infty} \mu(A_n) < \infty$;
- II $S(x) = \sum_{n=1}^{\infty} \alpha_n A_n(x)$ for every $x \in X$ for which $\sum_{n=1}^{\infty} A_n(x) < \infty$.

If conditions I and II are satisfied, we write $S \simeq \sum_{n=1}^{\infty} \alpha_n A_n$.

It can be shown that, if $S \simeq \sum_{n=1}^{\infty} \alpha_n A_n$ and $S \simeq \sum_{n=1}^{\infty} \beta_n B_n$, then $\sum_{n=1}^{\infty} \alpha_n \mu(A_n) = \sum_{n=1}^{\infty} \beta_n \mu(B_n)$. This enables us to define an extension of μ to all summable sets:

If $S \simeq \sum_{n=1}^{\infty} \alpha_n A_n$, then $\mu(S) = \sum_{n=1}^{\infty} \alpha_n \mu(A_n)$.

A set $S \subset X$ is called μ -measurable if the set $S \cap A$ is μ -summable for every $A \in \mathcal{R}$. Let $\Sigma(\mathcal{R}, \mu)$ denote the collection of all μ -measurable subsets of X . If $S \subset X$ is a μ -measurable set that is not μ -summable, then we define $\mu(S) = \infty$. It can be shown that $\Sigma(\mathcal{R}, \mu)$ is a σ -algebra containing \mathcal{R} and $(X, \Sigma(\mathcal{R}, \mu), \mu)$ is a complete measure space.

The following lemma will be needed in the proof of a result in the last section. We denote by μ_A the restriction of μ to A , that is, $\mu_A(S) = \mu(S \cap A)$.

Lemma 2.1. *Let μ_1, \dots, μ_n be finite measures on (X, Σ) . Then there exists a measure μ on (X, Σ) such that for every $\varepsilon > 0$ there is a finite partition π of X such that for $i = 1, \dots, n$ we have*

$$\mu_i = \sum_{S \in \pi} \alpha_{i,S} \mu_S + \kappa_i$$

where $\alpha_{i,S} \geq 0$ and $\kappa_1, \dots, \kappa_n$ are measures on (X, Σ) such that

$$(2.1) \quad \kappa_1(X) + \dots + \kappa_n(X) < \varepsilon.$$

In particular, we may define μ to be $\sum_{i=1}^n \mu_i$.

Proof. Let $\varepsilon > 0$. Notice that the measures μ_1, \dots, μ_n are absolutely continuous with respect to μ as defined above. Consider the Radon-Nikodym derivatives f_i of μ_i , that is,

$$\mu_i(B) = \int_B f_i d\mu$$

for all $B \in \Sigma$. Since each f_i is a non-negative and integrable with respect to μ , there are simple functions $\sum_{A \in \pi_i} \alpha_{i,A} A(x)$, with respect

to finite partitions π_i of X , such that $\alpha_{i,A} \geq 0$, $\sum_{A \in \pi_i} \alpha_{i,A} A(x) \leq f_i$, and

$$\int_X \left(f_i - \sum_{A \in \pi_i} \alpha_{i,A} A(x) \right) d\mu < \frac{\varepsilon}{n}.$$

Now define the common refinement of the partitions π_i to be π , and define the measures κ_i by the equation

$$\kappa_i(B) = \int_B \left(f_i - \sum_{A \in \pi_i} \alpha_{i,A} \right) d\mu$$

for all $B \in \Sigma$. Notice that each simple function with respect to π_i is also a simple function with respect to π , that is,

$$\sum_{A \in \pi_i} \alpha_{i,A} A(x) = \sum_{S \in \pi} \alpha_{i,S} S(x)$$

where $\alpha_{i,A} = \alpha_{i,S}$ if $S \subseteq A$. Consequently, for every $B \in \Sigma$, we have

$$\begin{aligned} \mu_i(B) &= \int_B \sum_{S \in \pi} \alpha_{i,S} S(x) d\mu + \int_B \left(f_i - \sum_{S \in \pi} \alpha_{i,S} S(x) \right) d\mu \\ &= \sum_{S \in \pi} \alpha_{i,S} \mu_S(B) + \kappa_i(B), \end{aligned}$$

and the κ_i 's were constructed to satisfy (2.1). \square

3. VECTOR MEASURES

Let X be a nonempty set, \mathcal{R} a ring of subsets of X , and let $(\mathbb{E}, \|\cdot\|)$ be a Banach space.

A set function $\mu : \mathcal{R} \rightarrow \mathbb{E}$ is called *σ -additive* if for any sequence of disjoint sets $A_1, A_2, \dots \in \mathcal{R}$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$ we have

$$\mu \left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Definition 3.1. By the *variation* of a σ -additive set function $\mu : \mathcal{R} \rightarrow \mathbb{E}$ we mean the set function $|\mu| : \mathcal{R} \rightarrow [0, \infty]$ defined by

$$|\mu|(A) = \sup \left\{ \sum_{B \in \pi} \|\mu(B)\| : \pi \subset \mathcal{R} \text{ is a finite partition of } A \right\}.$$

Note that $|\mu|(A) \geq \|\mu(A)\|$ for any set $A \in \mathcal{R}$. If $|\mu|(X) < \infty$, then we say that μ is of *bounded variation*.

In the remainder of this section we assume that $\mu : \mathcal{R} \rightarrow \mathbb{E}$ is a σ -additive set function of bounded variation.

Lemma 3.2. $|\mu|$ is monotone and finitely additive.

Proof. Clearly $|\mu|$ is monotone. Now consider disjoint $A_1, \dots, A_n \in \mathcal{R}$. Since the union of finite partitions of A_1, \dots, A_n is a finite partition of $A_1 \cup \dots \cup A_n$, we have

$$|\mu| \left(\bigcup_{k=1}^n A_k \right) \geq \sum_{k=1}^n |\mu|(A_k).$$

On the other hand, every finite partition π of $\bigcup_{k=1}^n A_k$ can be refined to a partition $\bigcup_{k=1}^n \pi_k$, where π_k is a finite partition of A_k for $k = 1, \dots, n$. Then

$$\sum_{B \in \pi} \|\mu(B)\| \leq \sum_{B \in \pi_1 \cup \dots \cup \pi_n} \|\mu(B)\| = \sum_{k=1}^n \sum_{B \in \pi_k} \|\mu(B)\| \leq \sum_{k=1}^n |\mu|(A_k)$$

and consequently

$$|\mu| \left(\bigcup_{k=1}^n A_k \right) \leq \sum_{k=1}^n |\mu|(A_k).$$

□

Theorem 3.3. $|\mu|$ is a σ -additive positive measure on \mathcal{R} .

Proof. Let $A_1, A_2, \dots \in \mathcal{R}$ be a sequence of disjoint sets such that $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$. By Lemma 3.2, for every $m \in \mathbb{N}$, we have

$$\sum_{n=1}^m |\mu|(A_n) \leq |\mu|(A).$$

Consequently

$$\sum_{n=1}^{\infty} |\mu|(A_n) \leq |\mu|(A).$$

Now let ε be an arbitrary positive number. Then

$$|\mu|(A) < \sum_{j=1}^k \|\mu(B_j)\| + \varepsilon,$$

for some partition $\{B_1, \dots, B_k\}$ of A , and we have

$$\begin{aligned}
|\mu|(A) &< \sum_{j=1}^k \|\mu(B_j)\| + \varepsilon \\
&= \sum_{j=1}^k \left\| \mu \left(\bigcup_{n=1}^{\infty} (A_n \cap B_j) \right) \right\| + \varepsilon \\
&\leq \sum_{j=1}^k \sum_{n=1}^{\infty} \|\mu(A_n \cap B_j)\| + \varepsilon \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^k \|\mu(A_n \cap B_j)\| + \varepsilon \\
&\leq \sum_{n=1}^{\infty} |\mu(A_n)| + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we obtain

$$|\mu|(A) \leq \sum_{n=1}^{\infty} |\mu(A_n)|.$$

□

Corollary 3.4. $|\mu|$ can be extended to a complete σ -additive measure on $\Sigma(\mathcal{R}, |\mu|)$.

We are going to use $|\mu|$ to denote the variation of μ defined on \mathcal{R} as well as its extension to a positive measure on $\Sigma(\mathcal{R}, |\mu|)$.

Our next goal is to define an extension of μ to a σ -additive \mathbb{E} -valued measure on $\Sigma(\mathcal{R}, |\mu|)$.

Lemma 3.5. If $\emptyset \simeq \sum_{n=1}^{\infty} \alpha_n A_n$ for some $A_n \in \mathcal{R}$ and $\alpha_n \in \{-1, 1\}$, then $\sum_{n=1}^{\infty} \alpha_n \mu(A_n) = 0$.

Proof. First note that, if $\emptyset \simeq \sum_{n=1}^{\infty} \alpha_n A_n$, then $\sum_{n=1}^{\infty} \alpha_n |\mu|(A_n) = 0$. Let $\varepsilon > 0$ and let $n_0 \in \mathbb{N}$ be such that

$$\sum_{n=n_0+1}^{\infty} |\mu|(A_n) < \varepsilon.$$

Let $C_1, \dots, C_{n_1}, D_1, \dots, D_{n_2} \in \mathcal{R}$ be such that $C_j \cap D_k = \emptyset$ for all j and k and such that

$$(3.1) \quad \alpha_1 A_1 + \dots + \alpha_{n_0} A_{n_0} = C_1 + \dots + C_{n_1} - D_1 - \dots - D_{n_2}.$$

Note that

$$\emptyset \simeq C_1 + \dots + C_{n_1} - D_1 - \dots - D_{n_2} + \alpha_{n_0+1} A_{n_0+1} + \alpha_{n_0+2} A_{n_0+2} + \dots$$

If $V = \text{supp}(D_1 \cup \dots \cup D_{n_2})$, then

$$\emptyset \simeq -D_1 - \dots - D_{n_2} + \alpha_{n_0+1}(A_{n_0+1} \cap V) + \alpha_{n_0+2}(A_{n_0+2} \cap V) + \dots$$

and hence

$$\begin{aligned} 0 &= -|\mu|(D_1) - \dots - |\mu|(D_{n_2}) + \alpha_{n_0+1}|\mu|(A_{n_0+1} \cap V) + \alpha_{n_0+2}|\mu|(A_{n_0+2} \cap V) + \dots \\ &< -|\mu|(D_1) - \dots - |\mu|(D_{n_2}) + \varepsilon. \end{aligned}$$

Consequently

$$(3.2) \quad |\mu|(D_1) + \dots + |\mu|(D_{n_2}) < \varepsilon$$

and since

$$0 = \sum_{n=1}^{n_1} |\mu|(C_n) - \sum_{n=1}^{n_2} |\mu|(D_n) + \sum_{n=n_0+1}^{\infty} \alpha_n |\mu|(A_n),$$

also

$$(3.3) \quad |\mu|(C_1) + \dots + |\mu|(C_{n_1}) < 2\varepsilon.$$

From (3.1), (3.2), and (3.3) we obtain

$$\begin{aligned} \left\| \sum_{n=1}^{n_0} \alpha_n \mu(A_n) \right\| &= \|\mu(C_1) + \dots + \mu(C_{n_1}) - \mu(D_1) - \dots - \mu(D_{n_2})\| \\ &\leq \|\mu(C_1)\| + \dots + \|\mu(C_{n_1})\| + \|\mu(D_1)\| + \dots + \|\mu(D_{n_2})\| \\ &\leq |\mu|(C_1) + \dots + |\mu|(C_{n_1}) + |\mu|(D_1) + \dots + |\mu|(D_{n_2}) \\ &< 3\varepsilon \end{aligned}$$

For any $m > n_0$ we have

$$\begin{aligned} \left\| \sum_{n=1}^m \alpha_n \mu(A_n) \right\| &\leq \left\| \sum_{n=1}^{n_0} \alpha_n \mu(A_n) \right\| + \left\| \sum_{n=n_0+1}^m \alpha_n \mu(A_n) \right\| \\ &\leq \left\| \sum_{n=1}^{n_0} \alpha_n \mu(A_n) \right\| + \sum_{n=n_0+1}^m \|\mu(A_n)\| \\ &\leq \sum_{n=1}^m |\mu|(A_n) < 4\varepsilon \end{aligned}$$

Since ε is arbitrary, we obtain $\sum_{n=1}^{\infty} \alpha_n \mu(A_n) = 0$. \square

Corollary 3.6. *If $S \simeq \sum_{n=1}^{\infty} \alpha_n A_n$ and $S \simeq \sum_{n=1}^{\infty} \beta_n B_n$ for some $A_n, B_n \in \mathcal{R}$ and $\alpha_n, \beta_n \in \{-1, 1\}$, then $\sum_{n=1}^{\infty} \alpha_n \mu(A_n) = \sum_{n=1}^{\infty} \beta_n \mu(B_n)$.*

Proof. It suffices to note that

$$\emptyset \simeq \alpha_1 A_1 - \beta_1 B_1 + \alpha_2 A_2 - \beta_2 B_2 + \dots$$

and then use Lemma 3.5. \square

The above corollary justifies the following definition of \mathbb{E} -valued measure on $\Sigma(\mathcal{R}, |\mu|)$.

Definition 3.7. If $S \simeq \sum_{n=1}^{\infty} \alpha_n A_n$ for some $A_n \in \mathcal{R}$ and $\alpha_n \in \{-1, 1\}$, then we define $\mu(S) = \sum_{n=1}^{\infty} \alpha_n \mu(A_n)$.

It remains to show that μ is σ -additive on $\Sigma(\mathcal{R}, |\mu|)$. If $U \subset X$, $U_1, U_2, \dots \in \Sigma(\mathcal{R}, |\mu|)$, and

- I $\sum_{n=1}^{\infty} |\mu|(U_n) < \infty$;
- II $U(x) = \sum_{n=1}^{\infty} \alpha_n U_n(x)$ for every $x \in X$ for which $\sum_{n=1}^{\infty} U_n(x) < \infty$,

then we write $U \simeq \sum_{n=1}^{\infty} \alpha_n U_n$.

Lemma 3.8. *If $U \simeq \sum_{n=1}^{\infty} \alpha_n U_n$ for some $U_n \in \Sigma(\mathcal{R}, |\mu|)$ and $\alpha_n \in \{-1, 1\}$, then $\mu(U) = \sum_{n=1}^{\infty} \alpha_n \mu(U_n)$.*

Proof. Since $U_1, U_2, \dots \in \Sigma(\mathcal{R}, |\mu|)$, for every $k \in \mathbb{N}$ there exists an expansion $U_k \simeq \sum_{n=1}^{\infty} \beta_{k,n} B_{k,n}$, with $B_{k,n} \in \mathcal{R}$ and such that

$$\sum_{n=1}^{\infty} |\mu|(B_{k,n}) < |\mu|(U_k) + 2^{-k}.$$

Let $\sum_{n=1}^{\infty} \gamma_n C_n$ be a series arranged from all series $\sum_{n=1}^{\infty} \alpha_k \beta_{k,n} B_{k,n}$. Then $U \simeq \sum_{n=1}^{\infty} \gamma_n C_n$ and

$$\mu(U) = \sum_{n=1}^{\infty} \gamma_n \mu(C_n) = \sum_{n=1}^{\infty} \alpha_n \mu(U_n).$$

□

Theorem 3.9. μ is σ -additive on $\Sigma(\mathcal{R}, |\mu|)$.

Proof. If $U_1, U_2, \dots \in \Sigma(\mathcal{R}, |\mu|)$ are disjoint, then $\bigcup_{n=1}^{\infty} U_n \simeq \sum_{n=1}^{\infty} U_n$. Therefore $\mu(\bigcup_{n=1}^{\infty} U_n) = \sum_{n=1}^{\infty} \mu(U_n)$, by Lemma 3.8. □

4. VECTOR MEASURES AND BOCHNER INTEGRABLE FUNCTIONS

Let (X, Σ, μ) be a measure space and let \mathbb{E} be a Banach space. A function $f : X \rightarrow \mathbb{E}$ is Bochner integrable [4] if there are sets $A_1, A_2, \dots \in \Sigma$ and vectors $v_1, v_2, \dots \in \mathbb{E}$ such that

- I $\sum_{n=1}^{\infty} \mu(A_n) < \infty$;
- II $f(x) = \sum_{n=1}^{\infty} v_n A_n(x)$ for every $x \in X$ for which $\sum_{n=1}^{\infty} \|v_n\| A_n(x) < \infty$.

If for some $f : X \rightarrow \mathbb{E}$, $A_1, A_2, \dots \in \Sigma$, and $v_1, v_2, \dots \in \mathbb{E}$ conditions I and II are satisfied, we will write $f \simeq \sum_{n=1}^{\infty} v_n A_n$.

If $f : X \rightarrow \mathbb{E}$ is a Bochner integrable function on a measure space (X, Σ, μ) , then

$$\omega(S) = \int_S f d\mu$$

defines a vector measure of bounded variation on (X, Σ) . Not every vector measure is of this form (see [1]). We say that a Banach space \mathbb{E} has the *Radon-Nikodym property* if for every vector measure ω on (X, Σ) of bounded variation that is absolutely continuous with respect to a measure μ on (X, Σ) there exists a Bochner integrable function f on (X, Σ, μ) such that $\omega(S) = \int_S f d\mu$ for every $S \in \Sigma$.

If μ_1, μ_2, \dots are positive measures on a measurable space (X, Σ) and $v_1, v_2, \dots \in \mathbb{E}$ are such that

$$\sum_{n=1}^{\infty} \|v_n\| \mu_n(X) < \infty,$$

then

$$\omega(S) = \sum_{n=1}^{\infty} v_n \mu_n(S)$$

defines a vector measure of bounded variation on (X, Σ) .

Theorem 4.1. *If \mathbb{E} has the Radon-Nikodym property, then every \mathbb{E} -valued measure ω on a measure space (X, Σ) is of the form*

$$\omega(S) = \sum_{n=1}^{\infty} v_n \mu_n(S)$$

where μ_1, μ_2, \dots are positive measures on (X, Σ) and $v_1, v_2, \dots \in \mathbb{E}$ are such that $\sum_{n=1}^{\infty} \|v_n\| \mu_n(X) < \infty$.

Proof. Let ω be a \mathbb{E} -valued measure of bounded variation on (X, Σ) . If \mathbb{E} has the Radon-Nikodym property, there exists a Bochner integrable function f on $(X, \Sigma, |\omega|)$ such that $\omega(S) = \int_S f d|\omega|$ for every $S \in \Sigma$. Let $A_1, A_2, \dots \in \Sigma$ and $v_1, v_2, \dots \in \mathbb{E}$ be such that $f \simeq \sum_{n=1}^{\infty} v_n \chi_{A_n}$. Then

$$\int_S f d|\omega| = \sum_{n=1}^{\infty} v_n |\omega|(S \cap A_n).$$

If we define

$$\mu_n(S) = |\omega|(S \cap A_n),$$

then

$$\omega(S) = \int_S f d|\omega| = \sum_{n=1}^{\infty} v_n \mu_n(S).$$

□

5. EXTENSION OF TRANSFUNCTIONS

In this section we define extensions of transfunctions to signed measures and vector measures. We are interested in properties of such extensions and the question of uniqueness.

By $\mathcal{M}(X, \Sigma, \mathbb{R})$ we denote the collection of all signed measures on (X, Σ) with bounded variation. For every $\mu \in \mathcal{M}(X, \Sigma, \mathbb{R})$ the Jordan decomposition theorem defines unique measures $\mu^+, \mu^- \in \mathcal{M}(X, \Sigma, \mathbb{R}^+)$ such that $\mu = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$. If $\Phi : \mathcal{M}(X, \Sigma_X, \mathbb{R}^+) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$ is a transfunction, then Φ has a natural extension to a transfunction $\tilde{\Phi} : \mathcal{M}(X, \Sigma_X, \mathbb{R}) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{R})$, namely

$$\tilde{\Phi}\mu = \Phi\mu^+ - \Phi\mu^-.$$

Clearly, there are other ways of extending Φ to a transfunction $\tilde{\Phi} : \mathcal{M}(X, \Sigma_X, \mathbb{R}) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{R})$. We show that the above extension has desirable properties and that under some additional conditions it is unique.

Lemma 5.1. *If Φ is strongly additive, then $\tilde{\Phi}(\mu_1 - \mu_2) = \Phi\mu_1 - \Phi\mu_2$ for any $\mu_1, \mu_2 \in \mathcal{M}(X, \Sigma, \mathbb{R}^+)$.*

Proposition 5.2. *Let $\Phi : \mathcal{M}(X, \Sigma_X, \mathbb{R}^+) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$ be a transfunction and let $\tilde{\Phi} : \mathcal{M}(X, \Sigma_X, \mathbb{R}) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{R})$ be the transfunction defined by $\tilde{\Phi}\mu = \Phi\mu^+ - \Phi\mu^-$.*

- (a) *If Φ is weakly additive, then $\tilde{\Phi}(\mu_1 + \mu_2) = \tilde{\Phi}\mu_1 + \tilde{\Phi}\mu_2$ and $\tilde{\Phi}(\mu_1 - \mu_2) = \tilde{\Phi}\mu_1 - \tilde{\Phi}\mu_2$ whenever $\mu_1 \perp \mu_2$.*
- (b) *If Φ is strongly additive, then so is $\tilde{\Phi}$.*
- (c) *If $\Phi(\alpha\mu) = \alpha\Phi(\mu)$ for any $\alpha > 0$, then $\tilde{\Phi}(\alpha\mu) = \alpha\tilde{\Phi}(\mu)$ for any $\alpha \in \mathbb{R}$.*
- (d) *If Φ is monotone and strongly additive, then $\tilde{\Phi}$ is monotone.*
- (e) *If Φ is norm preserving, then so is $\tilde{\Phi}$.*
- (f) *If $\|\Phi(\mu)\| \leq C\|\mu\|$ for some $C > 0$ and all $\mu \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$, then $\|\tilde{\Phi}(\mu)\| \leq C\|\mu\|$ for all $\mu \in \mathcal{M}(X, \Sigma_X, \mathbb{R})$.*
- (g) *If Φ is continuous with respect to weak convergence, then so is $\tilde{\Phi}$.*
- (h) *If Φ is continuous with respect to uniform convergence, then so is $\tilde{\Phi}$.*

Proposition 5.3. *If $\Phi : \mathcal{M}(X, \Sigma_X, \mathbb{R}^+) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$ is a strongly additive transfunction, then*

$$\tilde{\Phi}\mu = \Phi\mu^+ - \Phi\mu^-.$$

is a unique extension of Φ to a strongly additive transfunction from $\mathcal{M}(X, \Sigma_X, \mathbb{R})$ to $\mathcal{M}(Y, \Sigma_Y, \mathbb{R})$.

Proof. Let Ψ be an arbitrary extension of Φ to a strongly additive transfunction from $\mathcal{M}(X, \Sigma_X, \mathbb{R})$ to $\mathcal{M}(Y, \Sigma_Y, \mathbb{R})$. Since

$$\Phi\mu^+ = \Psi\mu^+ = \Psi(\mu^+ - \mu^- + \mu^-) = \Psi(\mu^+ - \mu^-) + \Psi\mu^- = \Psi(\mu^+ - \mu^-) + \Phi\mu^-,$$

we have $\Psi(\mu^+ - \mu^-) = \Phi\mu^+ - \Phi\mu^-$. \square

Proposition 5.4. *If $\Phi : \mathcal{M}(X, \Sigma_X, \mathbb{R}^+) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$ is a weakly additive transfunction, then*

$$\tilde{\Phi}\mu = \Phi\mu^+ - \Phi\mu^-.$$

is a unique extension of Φ to a weakly additive transfunction from $\mathcal{M}(X, \Sigma_X, \mathbb{R})$ to $\mathcal{M}(Y, \Sigma_Y, \mathbb{R})$ such that $\tilde{\Phi}(-\mu) = -\tilde{\Phi}\mu$ for all $\mu \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$.

Proof. If Ψ is an arbitrary extension of Φ to a weakly additive transfunction from $\mathcal{M}(X, \Sigma_X, \mathbb{R})$ to $\mathcal{M}(Y, \Sigma_Y, \mathbb{R})$ such that $\Psi(-\mu) = -\tilde{\Phi}\mu$

for all $\mu \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$, then

$$\Psi(\mu^+ - \mu^-) = \Psi(\mu^+) + \Psi(-\mu^-) = \Psi\mu^+ - \Psi\mu^- = \Phi\mu^+ - \Phi\mu^-.$$

□

Now we turn our attention to extensions of transfunctions to vector measures. Since, in general, \mathbb{R} is not a subspace of a vector space, formally we cannot talk about an extension of $\Phi : \mathcal{M}(X, \Sigma_X, \mathbb{R}^+) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$ to vector measures. On the other hand, we are interested in constructing a transfunction $\tilde{\Phi} : \mathcal{M}(X, \Sigma_X, \mathbb{E}) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{E})$, where \mathbb{E} is a Banach space, such that $\tilde{\Phi}(v\mu) = v\Phi\mu$ for every $\mu \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$ and $v \in \mathbb{E}$. So in some sense $\tilde{\Phi}$ is an “extension” of Φ .

For a bounded transfunction $\Phi : \mathcal{M}(X, \Sigma_X, \mathbb{R}) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{R})$ we define

$$\|\Phi\| = \inf\{C : \|\Phi(\mu)\| \leq C\|\mu\| \text{ for all } \mu \in \mathcal{M}(X, \Sigma_X, \mathbb{R})\},$$

so that we have $\|\Phi(\mu)\| \leq \|\Phi\|\|\mu\|$ for all $\mu \in \mathcal{M}(X, \Sigma_X, \mathbb{R})$.

We begin with a technical lemma.

Lemma 5.5. *Let $\Phi : \mathcal{M}(X, \Sigma_X, \mathbb{R}^+) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$ be a bounded, strongly additive, and homogeneous transfunction and let \mathbb{E} be a Banach space. Then*

$$\|v_1\Phi\mu_1 + \cdots + v_n\Phi\mu_n\| \leq \|\Phi\|\|v_1\mu_1 + \cdots + v_n\mu_n\|$$

for all $v_1, \dots, v_n \in \mathbb{E}$, $\mu_1, \dots, \mu_n \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$, and $n \in \mathbb{N}$.

Proof. Let $v_1, \dots, v_n \in \mathbb{E}$, $\mu_1, \dots, \mu_n \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$, and let $\mu = \sum_{k=1}^n \mu_k$. By Lemma 2.1, for any $\varepsilon > 0$ there are disjoint sets $S_1, \dots, S_N \in \Sigma$ such that for $i = 1, \dots, n$ we have

$$\mu_i = \sum_{j=1}^N \alpha_{i,j} \mu_{S_j} + \kappa_i,$$

where $\alpha_{k,j} \geq 0$ and $\kappa_1, \dots, \kappa_n$ are measures such that

$$\|v_1\kappa_1 + \cdots + v_n\kappa_n\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|v_1\Phi(\kappa_1) + \cdots + v_n\Phi(\kappa_n)\| < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned}
\|v_1\Phi(\mu_1) + \cdots + v_n\Phi(\mu_n)\| &= \left\| \sum_{i=1}^n v_i \Phi \left(\sum_{j=1}^N \alpha_{i,j} \mu_{S_j} + \kappa_i \right) \right\| \\
&= \left\| \sum_{i=1}^n v_i \left(\sum_{j=1}^N \alpha_{i,j} \Phi(\mu_{S_j}) + \Phi(\kappa_i) \right) \right\| \\
&= \left\| \sum_{j=1}^N \left(\sum_{i=1}^n \alpha_{i,j} v_i \right) \Phi(\mu_{S_j}) + \sum_{i=1}^n v_i \Phi(\kappa_i) \right\| \\
&\leq \left\| \sum_{j=1}^N \left(\sum_{i=1}^n \alpha_{i,j} v_i \right) \Phi(\mu_{S_j}) \right\| + \left\| \sum_{i=1}^n v_i \Phi(\kappa_i) \right\| \\
&\leq \left\| \sum_{j=1}^N \left(\sum_{i=1}^n \alpha_{i,j} v_i \right) \Phi(\mu_{S_j}) \right\| + \frac{\varepsilon}{2} \\
&\leq \sum_{j=1}^N \left\| \left(\sum_{i=1}^n \alpha_{i,j} v_i \right) \Phi(\mu_{S_j}) \right\| + \frac{\varepsilon}{2} \\
&\leq \sum_{j=1}^N \left\| \sum_{i=1}^n \alpha_{i,j} v_i \right\| \|\Phi(\mu_{S_j})\| + \frac{\varepsilon}{2} \\
&\leq \|\Phi\| \sum_{j=1}^N \left\| \sum_{i=1}^n \alpha_{i,j} v_i \right\| \|\mu_{S_j}\| + \frac{\varepsilon}{2}.
\end{aligned}$$

Since

$$\begin{aligned}
\|v_1\mu_1 + \cdots + v_n\mu_n\| &= \left\| \sum_{i=1}^n v_i \sum_{j=1}^N \alpha_{i,j} \mu_{S_j} + \sum_{i=1}^n v_i \kappa_i \right\| \\
&\geq \left\| \sum_{i=1}^n v_i \sum_{j=1}^N \alpha_{i,j} \mu_{S_j} \right\| - \frac{\varepsilon}{2} \\
&= \left\| \sum_{j=1}^N \sum_{i=1}^n \alpha_{i,j} v_i \mu_{S_j} \right\| - \frac{\varepsilon}{2} \\
&= \sum_{j=1}^N \left\| \sum_{i=1}^n \alpha_{i,j} v_i \mu_{S_j} \right\| - \frac{\varepsilon}{2} \\
&= \sum_{j=1}^N \left\| \sum_{i=1}^n \alpha_{i,j} v_i \right\| \|\mu_{S_j}\| - \frac{\varepsilon}{2},
\end{aligned}$$

we get

$$\|v_1\Phi(\mu_1) + \cdots + v_n\Phi(\mu_n)\| \leq \|v_1\mu_1 + \cdots + v_n\mu_n\| + \varepsilon.$$

Since ε is an arbitrary positive number, the desired inequality follows. \square

Corollary 5.6. *Let $\Phi : \mathcal{M}(X, \Sigma_X, \mathbb{R}^+) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$ be a bounded, strongly additive, and homogeneous transfunction and let \mathbb{E} be a Banach space. If*

$$\sum_{n=1}^{\infty} v_n \mu_n = 0,$$

for some $v_n \in \mathbb{E}$ and $\mu_n \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$, then

$$\sum_{n=1}^{\infty} v_n \Phi \mu_n = 0.$$

Proof. Since

$$\left\| \sum_{j=1}^n v_j \Phi \mu_j \right\| \leq \|\Phi\| \left\| \sum_{j=1}^n v_j \mu_j \right\|,$$

we have

$$\left\| \sum_{j=1}^{\infty} v_j \Phi \mu_j \right\| \leq \|\Phi\| \left\| \sum_{j=1}^{\infty} v_j \mu_j \right\| = 0.$$

\square

In the next theorem we show that bounded, strongly additive, and homogeneous transfunctions can be extended to vector measures of a special type, namely measures that can be defined as sums of series of positive measures multiplied by elements of a Banach space \mathbb{E} . We will denote this space of measures by $\mathcal{M}_s(X, \Sigma_X, \mathbb{E})$, that is,

$$\mathcal{M}_s(X, \Sigma_X, \mathbb{E}) = \left\{ \sum_{n=1}^{\infty} v_n \mu_n : \mu_n \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+), v_n \in \mathbb{E}, \sum_{n=1}^{\infty} \|v_n\| \|\mu_n\| < \infty \right\}.$$

Theorem 5.7. *Let $\Phi : \mathcal{M}(X, \Sigma_X, \mathbb{R}^+) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$ be a bounded, strongly additive, and homogeneous transfunction and let \mathbb{E} be a Banach space. Then there is a unique bounded linear transfunction $\tilde{\Phi} : \mathcal{M}_s(X, \Sigma_X, \mathbb{E}) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{E})$ satisfying $\tilde{\Phi}(v\mu) = v\Phi\mu$ for every $\mu \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$ and every $v \in \mathbb{E}$.*

Proof. Let $\Phi : \mathcal{M}(X, \Sigma_X, \mathbb{R}^+) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$ be a bounded, strongly additive, and homogeneous transfunction and let \mathbb{E} be a Banach space.

If $\mu = \sum_{n=1}^{\infty} v_n \mu_n$, for some $v_n \in \mathbb{E}$ and $\mu_n \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$ such that $\sum_{n=1}^{\infty} \|v_n\| \|\mu_n\| < \infty$, then we define

$$\tilde{\Phi}(\mu) = \sum_{n=1}^{\infty} v_n \Phi \mu_n.$$

Since

$$\sum_{n=1}^{\infty} \|v_n \Phi \mu_n\| \leq \sum_{n=1}^{\infty} \|v_n\| \|\Phi \mu_n\| \leq \|\Phi\| \sum_{n=1}^{\infty} \|v_n\| \|\mu_n\| < \infty,$$

the series converges. Moreover, if $\sum_{n=1}^{\infty} v_n \mu_n = \sum_{n=1}^{\infty} w_n \kappa_n$, for some $v_n, w_n \in \mathbb{E}$ and $\mu_n, \kappa_n \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$ such that $\sum_{n=1}^{\infty} \|v_n\| \|\mu_n\| < \infty$ and $\sum_{n=1}^{\infty} \|w_n\| \|\kappa_n\| < \infty$, then

$$v_1 \mu_1 - w_1 \kappa_1 + v_2 \mu_2 - w_2 \kappa_2 + \cdots = 0.$$

By Corollary 5.6,

$$v_1 \Phi \mu_1 - w_1 \Phi \kappa_1 + v_2 \Phi \mu_2 - w_2 \Phi \kappa_2 + \cdots = 0,$$

so $\sum_{n=1}^{\infty} v_n \Phi \mu_n = \sum_{n=1}^{\infty} w_n \Phi \kappa_n$. This shows that the extension is well-defined.

Clearly, $\tilde{\Phi}$ is a linear transfunction from $\mathcal{M}_s(X, \Sigma_X, \mathbb{E})$ to $\mathcal{M}(Y, \Sigma_Y, \mathbb{E})$. Since, by Lemma 5.5, for every $n \in \mathbb{N}$ we have

$$\|v_1 \Phi \mu_1 + \cdots + v_n \Phi \mu_n\| \leq \|\Phi\| \|v_1 \mu_1 + \cdots + v_n \mu_n\|,$$

we have $\|\tilde{\Phi}(\mu)\| \leq \|\Phi\| \|\mu\|$, so $\tilde{\Phi}$ is bounded.

Now let $\Psi : \mathcal{M}_s(X, \Sigma_X, \mathbb{E}) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{E})$ be a bounded linear transfunction satisfying $\Psi(v\mu) = v\Psi\mu$ for every $\mu \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$ and every $v \in \mathbb{E}$. If $\mu = \sum_{n=1}^{\infty} v_n \mu_n$, for some $v_n \in \mathbb{E}$ and $\mu_n \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$ such that $\sum_{n=1}^{\infty} \|v_n\| \|\mu_n\| < \infty$, then for every $n \in \mathbb{N}$ we have

$$\Psi(v_1 \mu_1 + \cdots + v_n \mu_n) = \tilde{\Phi}(v_1 \mu_1 + \cdots + v_n \mu_n)$$

and consequently $\tilde{\Phi} = \Psi$ by continuity. \square

From the above theorem and Theorem 4.1 we obtain the following result.

Corollary 5.8. *Let \mathbb{E} be a Banach space with the Radon-Nikodym property. For every bounded, strongly additive, and homogeneous $\Phi : \mathcal{M}(X, \Sigma_X, \mathbb{R}^+) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{R}^+)$ there is a unique bounded linear transfunction $\tilde{\Phi} : \mathcal{M}(X, \Sigma_X, \mathbb{E}) \rightarrow \mathcal{M}(Y, \Sigma_Y, \mathbb{E})$ satisfying $\tilde{\Phi}(v\mu) = v\Phi\mu$ for every $\mu \in \mathcal{M}(X, \Sigma_X, \mathbb{R}^+)$ and every $v \in \mathbb{E}$.*

Since in the extension defined in Theorem 5.7 we assume that Φ is a bounded, strongly additive, and homogeneous transfunction and show that $\tilde{\Phi}$ is a bounded linear transfunction, of the properties discussed in Proposition 5.2 only the norm preserving property is not covered. An extension of a norm preserving transfunction need not be norm preserving.

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